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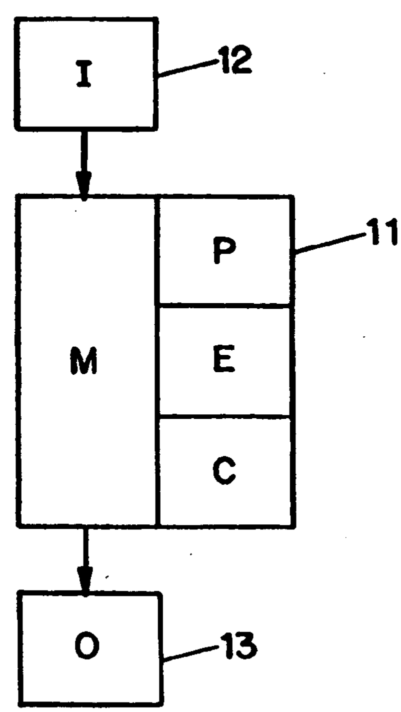
(54) Title: ESTIMATION METHOD AND SYSTEM FOR FINANCIAL SECURITIES TRADING

(57) Abstract

In setting the initial offering price of a financial instrument for purposes of securities trading, or in later revaluation as economic factors, such as interest rates, may change, an estimate of the value of the instrument may be represented as a multi-dimensional integral. For evaluation of the integral, numerical integration is preferred with the integrand being sampled at deterministic points having a low-discrepancy property. The technique produces approximate values at significant computational savings and with greater reliability as compared with the Monte-Carlo technique (11).



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Description

Estimation Method And System For Financial Securities Trading

The United States Government has certain rights to this invention pursuant to grants CCR-91-14042 and IRI-92-12597 awarded by the National Science Foundation, and to grant AFOSR-91-0347 awarded by the U.S. Air Force.

Background of the Invention

The invention relates to financial securities trading such as, e.g., trading in stocks, bonds and financial derivative instruments, including futures, options and collateralized mortgage obligations.

In financial securities trading, which includes the initial offer for sale, the value of a security may be estimated, e.g., based on expected future cash flow. Such cash flow may depend on variable interest rates, for example, and these and other relevant variables may be viewed as stochastic variables.

It is well known that the value of a financial security which depends on stochastic variables can be estimated in terms of a multi-dimensional integral. The dimension of such an integral may be very high.

In collateralized mortgage obligations (CMO), for example, instruments or securities variously called tranches, shares, participations, classes or contracts have cash flows which are determined by dividing and distributing the cash flow of an underlying collection or pool of mortgages on a monthly basis according to pre-specified rules. The present value of a tranche can be estimated on the basis of the expected monthly cash flows over the remaining term of the tranche, and

an estimate of the present value of a tranche can be r presented as a multi-dimensional integral whose dimension is the number of payment periods of the tranche. For a typical instrument with a 30-year term and with monthly payments, this dimension is 360.

Usually, such a high-dimensional integral can be evaluated only approximately, by numerical integration. This involves the generation of points in the domain of integration, evaluating or "sampling" the integrand at the generated points, and combining the resulting integrand values, e.g., by averaging. Well known for numerical integration in securities trading is the so-called Monte Carlo method in which points in the domain of integration are generated at random.

With integrands arising in financial securities trading, the computational work in combining the sampled values is negligible as compared with producing the integrand values. Thus, numerical integration methods in securities trading may be compared based on the number of samples required for obtaining a sufficiently accurate approximation to the integral.

Summary Of The Invention

A preferred method for estimating the value of a financial security involves numerical integration unlike Monte Carlo integration in that an integrand is sampled at deterministic points having a low-discrepancy property. As compared with the Monte Carlo method, significant advantages are realized with respect to speed, accuracy, and dependability.

Brief Description Of The Drawing

Fig. 1 is a schematic of a programmed computer system in accordance with a preferred embodiment of the invention.

Fig. 2 is a graphic representation of performance data obtained in computer trial runs with an exemplary embodiment of the invention as compared with two Monte Carlo computations.

Further included is an Appendix with two computer algorithms in "C" source language, respectively for computing Sobol points and Halton points. For a description of C, see B.W. Kernighan et al., The Programming Language C, Prentice-Hall, 1978.

Detailed Description Of Preferred Embodiments

Fig. 1 shows a stored-program computer 11 connected to input means 12, e.g., a keyboard, for entering financial securities data, and connected to output means 13, e.g., a visual display device, for displaying an estimated value of the financial security. The computer 11 includes a working memory M, a low-discrepancy deterministic point generator P, an integrand evaluator E, and an integrand-value combiner C.

In estimating the value of a multi-dimensional integral in financial securities trading, a multivariate integrand is sampled at points corresponding to a low-discrepancy deterministic sequence of points in the multivariate unit cube as defined below. If the multivariate unit cube is also the domain of integration, the points of the low-discrepancy deterministic sequence can be used as sample points directly. In the case of a more general domain of integration, sample points correspond to points of a low-discrepancy deterministic sequence in the multivariate unit cube via a suitable transformation or mapping.

When a sufficiently large number of sampled values has been computed, an approximation of the integral is obtained by suitably combining the computed values, e.g., by averaging or weighted averaging.

In the d-dimensional unit cube $D = [0,1]^d$, a low-discrepancy deterministic s quence $z_1, z_2, \ldots$ of points in D can be characterized as follows:

For a point $t = [t_1, \ldots, t_d]$ in D, define

$$[0, t) = [0, t_1) \times \ldots \times [0, t_d),$$

where $[0, t_i)$ denotes a left-closed, right-open interval, and denote with $\chi_{[0,t)}(.)$ the characteristic or indicator function of $[0, t)$. For points $z_1, \ldots, z_n$ in D, define

$$R_n(t;\ z_1, \ldots, z_n) = (\Sigma_{k=1}^{n} \chi_{[0,t)}(z_k))/n - t_1 t_2 \ldots t_d,$$

and define the discrepancy of $z_1, \ldots, z_n$ as the L_∞-norm of the function $R_n(.;\ z_1, \ldots, z_n)$, i.e.,

$$\|R_n(.;\ z_1, \ldots, z_n)\|_\infty = \sup_{t \in D} |R_n(t;\ z_1, \ldots, z_n)|.$$

The sequence $z_1, z_2, \ldots$ is said to be a low-discrepancy deterministic sequence provided

$$\|R_n(.;\ z_1, \ldots, z_n)\|_\infty = O((\log n)^d/n).$$

Low-discrepancy deterministic sequences are described in the published literature; see, e.g., H. Niederreiter, "Random Number Generation and Quasi-Monte Carlo Methods", CBMS-NSF, 63, SIAM, Philadelphia, 1992 and S. Paskov, "Average Case Complexity of Multivariate Integration for Smooth Functions", Journal of Complexity, Vol. 9 (1993), pp. 291-312. Well-known examples of low-discrepancy deterministic sequences are the so-called Hammersley points, Halton points, Sobol points, and hyperbolic-cross points.

For illustration, in the case of Sobol points in a single dimension (d=1), a constructive definition may be given as follows: Choose a primitive polynomial

$$P(x) = x^n + a_1 x^{n-1} + \ldots + a_{n-1} x + 1$$

(whose coefficients a_i are either 0 or 1) and define so-called direction numbers v_i, $i > n$ by the following recurrence formula:

$$v_i = a_1 v_{i-1} \oplus a_2 v_{i-2} \oplus \ldots \oplus a_{n-1} v_{i-n+1} \oplus v_{i-n} \oplus (v_{i-n}/2^n),$$

where $\oplus$ denotes a bit-by-bit "exclusive or" operation.

The initial numbers $v_1 = m_1/2, \ldots, v_n = m_n/2^n$ can be chosen freely provided m_i is odd and $0 < m_i < 2^i$ for $i = 1, 2, \ldots, n$.

Using the direction numbers v_i so defined, now define the one-dimensional Sobol sequence $x_1, x_2, \ldots$ by

$$x_k = b_1 v_1 \oplus b_2 v_2 \oplus \ldots \oplus b_w v_w, \quad k \geq 0$$

where $k = \Sigma_{i=0}^{\lceil \log k \rceil} b_i 2^i$ is the binary representation of k.

For higher dimensions (d > 1), the first d primitive polynomials $P_1, P_2, \ldots, P_d$ are used. If $\{x_k^i\}_{k=1}^{\infty}$ denotes the one-dimensional Sobol sequence generated by the polynomial P_i, the d-dimensional Sobol points are defined as $x_k = (x_k^1, x_k^2, \ldots, x_k^d)$.

While this definition can be implemented as a computer algorithm directly, faster techniques are known which produce these points in a "shuffled" or permuted sequence. In particular, this applies to the computer algorithm given in the Appendix.

For specificity in the following, a preferred embodiment of the invention is described as applied to a collateralized mortgage obligation known as CMO FN, 89-23. This has thirty-year maturity and consists of the following tranches:

PAC tranches 23-A, 23-B, 23-C, 23-D, 23-E
supporting tranches 23-G, 23-H, 23-J
residual tranche 23-R
accrual tranche 23-Z

The monthly cash flow is divided and distributed according to pre-specified rules which are included in a formal prospectus. Some of the basic rules may be stated as follows:

First from the monthly cash flow, the coupon is paid to the tranches. The remaining amount, called

Principal Distribution Amount, is used for repayment of the principal. Prior to a fixed date in the future, the Principal Amount will be allocated s quentially to the tranches 23-G, 23-H, 23-J and 23-Z. After that date, the Principal Distribution Amount will be allocated sequentially to the tranches 23-A, 23-B, 23-C, 23-D and 23-E according to a planned schedule. Any excess amount of the Principal Distribution Amount over the planned schedule will be allocated sequentially to the tranches 23-G, 23-H, 23-J and 23-Z. A distribution of principal of the tranche 23-R will be made only after all other tranches have been retired.

In deriving an estimate for the present value of the security at the time of issue, the following notation is used below:

C - the monthly payment on the underlying mortgage pool;

i_k - the projected interest rate in month k, $k = 1, 2, \ldots, 360$;

w_k - the percentage of mortgages prepaying in month k;

$a_{360-k+1}$ - the remaining annuity after month k.

A remaining annuity a_k can be expressed as3 to

$$a_k = 1 + v_0 + \ldots + v_0^{k-1}, \quad k = 1, 2, \ldots, 360,$$

with $v_0 = 1/(1+i_0)$, where i_0 is the current monthly interest rate. Thus, after k months, the remaining amount of principal borrowed is $C \cdot a_k$.

It is assumed that the interest rate i_k can be expressed as

$$i_k = K_0 \exp(\xi_k) i_{k-1},$$

where exp(.) denotes exponentiation and where $\xi_1, \xi_2, \ldots, \xi_{360}$ are independent, normally distributed random variables with mean 0 and variance σ, and K_0 is a given constant. For the present example, $\sigma = 0.0004$ is chosen.

It is assumed further that w_k as a function of i_k can be comput d as

$$w_k = K_1 + K_2 \arctan (K_3 i_k + K_4),$$

where K_1, K_2, K_3, K_4 are given constants.

Under these assumptions. the cash flow in month k, k = 1, 2, ..., 360 is

$$C(1-w_1(\xi_1))\ldots(1-w_{k-1}(\xi_1,\ldots,\xi_{k-1}))(1-w_k(\xi_1,\ldots,\xi_k) + w_k(\xi_1,\ldots,\xi_k)a_{360-k+1}),$$

where

$$w_k(\xi_1,\ldots,\xi_k) = K_1 + K_2 \arctan(K_3 K_0^k i_0 \exp(\xi_1+\ldots+\xi_k)+K_4).$$

This cash flow is distributed according to the rules of FN, 89-23. Then, the cash flow for each of the tranches is multiplied by the discount factor

$$v_1(\xi_1)\ldots v_k(\xi_1,\ldots,\xi_k), \quad \text{with}$$

$$v_j(\xi_1,\ldots,\xi_j) = 1/(1 + K_0^j i_0 \exp(\xi_1+\ldots+\xi_j)),\ j=1,2,\ldots,360,$$

to find the present value for month k. Summing of the present values for every month gives the present value PV_T, for each tranche T.

The expected or estimated value,

$$E(PV_T) = E(PV_T(\xi_1,\ldots,\xi_{360})),$$

upon a change of variables is represented by

$$E(PV_T) = \int_D PV_T(y_1(x_1),\ldots,y_{360}(x_{360}))dx_1\ldots dx_{360},$$

where

$$x_i = (2\pi\sigma)^{-1/2}\int_{-\infty}^{y_i} \exp(-t^2/(2\sigma))dt.$$

Thus, for each tranche T, a 360-variate integrand has to be integrated over the 360-dimensional unit cube, .

After generating a point

$$(x_1, x_2, \ldots, x_{360})$$

of a low-discrepancy deterministic sequence in the unit cube, the point

$$(y_1, y_2, \ldots, y_{360})$$

is computed by finding the value of the inverse normal cumulative distribution function at each x_k. Then, for each tranche T, the function value

$$PV_T(y_1, y_2, \ldots, y_{360})$$

is computed. These are the function values used in numerical integration.

Fig. 2 shows results from trial runs for CMO FN, 89-23 with a preferred method using Sobol points generated by the corresponding computer algorithm given in the Appendix, as compared with Monte Carlo integration. Two Monte Carlo computations were carried out, with different "seeds" or starting values of a congruential pseudo-random number generator known as RAN2; see W. Press et al., Numerical Recipes in C, Cambridge University Press, 1992. It is apparent that the preferred method reaches a steady value more rapidly. In this and other trial runs, with typical requirements of precision and confidence, a speed-up by a factor of 3 to 5 was realized as compared with Monte Carlo integration. Much greater speed-up can be expected when higher precision or/and higher levels of confidence are sought.

In a further trial run with CMO FN, 89-23, instead of Sobol points, Halton points were used as generated by the corresponding computer algorithm given in the Appendix. It is felt that Sobol points may be preferred over Halton points for integrals of high dimension. However, this preference may not apply in the case of lower-dimensional integrals, e.g., with dimension up to 5 or so.

A computation as described may be terminated after a predetermined number of function evaluations. Alternatively, e.g., after every function evaluation or after a predetermined incremental number of function evaluations, a current approximation may be compared with one or several preceding approximations, for termination once a suitable condition depending on the difference betw en approximations is met. Such termination criteria may be called automatic. Automatic

termination is particularly reliable where a sequence of approximations settles down smoothly; see, e.g., the curve in Fig. 2 corresponding to Sobol points.

Advantageously in computing function values, a cluster or network of multiple parallel processors or workstations can be used. This may involve a master or host processor providing points of a low-discrepancy sequence to slave processors and combining function values returned by the slave processors into an approximate value for the integral. Thus, the computation can be speeded up in proportion to the number of processors used.

Advantageous further, in combination with numerical integration as described above, is the use of error reduction techniques analogous to variance reduction in Monte Carlo integration as described, e.g., by M. Kalos et al., Monte Carlo Methods, John Wiley & Sons, 1986. This may involve a change of variables or/and variation reduction, for example.

```
/*******************************************************************

  void sobol(int n)

   This is the function sobol for generating Sobol points.  It returns the
   n-th d-dimensional Sobol point.  (The point (0,0, ..., 0) is skipped.)
   The point is implicitly returned through the array x.  The function
   sobseq  from  Numerical Recipes, 1992  was used as a basis, but
   changed significantly to accomodate the parallel distributed approach.
   The function  sobol  can generate Sobol points skipping an initial
   part of the sequence.  The constant MAXDIM, see  Numerical Recipes,
   is extended to 360 which required adding more initializing data to
   the arrays ip (the imitive polynomials),
   mdeg (their degrees), and iv (the initial direction numbers).  The
   polynomial x is used to generate the first coordinate of Sobol points.

   *****************************************************************/

#include "nrutil.h"
#define MAXBIT 30
#define MAXDIM 360


extern int d;    /* actual dimension of the points */
extern double *x;     /* This returns implicitly the n-th Sobol point in x */

void sobol(int n)
{
  int j,k,l;
  unsigned long i,im,ipp;
  static double fac;
  static unsigned long in,ix[MAXDIM+1],*iu[MAXBIT+1];
  static unsigned long mdeg[MAXDIM+1]={0,MAXBIT,1,2,3,3,4,4,
                          5,5,5,5,5,5,
                          6,6,6,6,6,6,
                          7,7,7,7,7,7,7,7,7,7,7,7,7,7,7,7,7,7,
/*16*/                          8,8,8,8,8,8,8,8,8,8,8,8,8,8,8,8,
                          9,9,9,9,9,9,9,9,9,9,9,9,9,9,9,9,
                          9,9,9,9,9,9,9,9,9,9,9,9,9,9,9,9,
                          9,9,9,9,9,9,9,9,9,9,9,9,9,9,9,9,
/*20*/          10,10,10,10,10,10,10,10,10,10,10,10,10,10,10,10,10,10,10,10,
         10,10,10,10,10,10,10,10,10,10,10,10,10,10,10,10,10,10,10,10,
         10,10,10,10,10,10,10,10,10,10,10,10,10,10,10,10,10,10,10,10,
         11,11,11,11,11,11,11,11,11,11,11,11,11,11,11,11,11,11,11,11,
         11,11,11,11,11,11,11,11,11,11,11,11,11,11,11,11,11,11,11,11,
         11,11,11,11,11,11,11,11,11,11,11,11,11,11,11,11,11,11,11,11,
         11,11,11,11,11,11,11,11,11,11,11,11,11,11,11,11,11,11,11,11,
         11,11,11,11,11,11,11,11,11,11,11,11,11,11,11,11,11,11,11,11,
         11,11,11,11,11,11,11,11,11,11,11,11,11,11,11,11,11,11,11,11,
         11,11,11,11,11,11,11,11,11,11,11,11,11,11,11,11,11,11,11,11,
         11,11,11,11,11,11,11,11,11,11,11,11,11,11,11,11,11,11,11,11,
         11,11,11,11,11,11,11,11,11,11,11,11,11,11,11,11,
         12,12,12,12,12,12,12,12,12,12,12,12,12,12,12,12,12,12,12,12,
         12,12,12
         };

 /* The first coordinate is set to MAXBIT, but it is never used */

 static unsigned long ip[MAXDIM+1]={0,MAXBIT,0,1,1,2,1,4,
```

```
/*deg 5*/                          2,13,7,14,11,4,
/*deg 6*/                          1,16,13,22,19,25,
/*deg 7*/                1,32,4,8,7,56,14,28,19,50,21,42,31,62,37,41,55,59,
/*deg 8*/           14,56,21,22,38,47,49,50,52,67,70,84,97,103,115,122,
/*deg 9*/  8,13 ,16 ,22 ,25,44 ,47 ,52 ,55 ,59 ,62 ,67 ,74 ,81 ,82 ,87 ,91 ,94,
     103,104,109,122,124,137,138,143,145,152,157,167,173,176,181,182,185,
     191,194,199,218,220,227,229,230,234,236,241,244,253,
/*deg 10*/     4,13,19,22,50,55,64,69,98,107,115,121,127,134,140,145,152,
      158,161,171,181,194,199,203,208,227,242,251,253,265,266,274,283,289,295,
      301,316,319,324,346,352,361,367,382,395,398,400,412,419,422,426,428,433,
      446,454,457,472,493,505,508,
/*deg11*/ 2,11,21,22,35,49,50,56,61,70,74,79,84,88,103,104,112,115,117,122,134,
      137,146,148,157,158,162,164,168,173,185,186,191,193,199,213,214,220,227,
      236,242,251,256,259,265,266,276,292,304,310,316,319,322,328,334,339,341,
      345,346,362,367,372,375,376,381,385,388,392,409,415,416,421,428,431,434,
      439,446,451,453,457,458,471,475,478,484,493,494,499,502,517,518,524,527,
      555,560,565,569,578,580,587,589,590,601,607,611,614,617,618,625,628,635,
      641,647,654,659,662,672,675,682,684,689,695,696,713,719,724,733,734,740,
      747,749,752,755,762,770,782,784,787,789,793,796,803,805,810,815,824,829,
      830,832,841,847,849,861,871,878,889,892,901,908,920,923,942,949,950,954,
      961,968,971,973,979,982,986,998,1001,1010,1012,
/*deg12*/  41,52,61,62,76,104,117,131,143,145,157,167,171,176,181,194,217,236,
     239,262,283,286,307};

  static unsigned long iv[MAXDIM*MAXBIT+1]={
    0,  /* 40 elements in a row */
1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,
1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,
1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,
1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,
1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,
1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,
1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,
1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,
1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,     /*end 1
*/

1,3,1,3,1,3,1,3,3,1,3,1,3,1,3,1,1,3,1,3,1,3,1,3,3,1,3,1,3,1,3,1,1,3,1,3,1,3,1,3,
1,3,1,3,1,3,1,3,3,1,3,1,3,1,3,1,1,3,1,3,1,3,1,3,3,1,3,1,3,1,3,1,1,3,1,3,1,3,1,3,
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9,    /*end 4 */

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3,19,29,11,19,27,15,25,  /*end 6 */

1,85,67,49,125,25,109,43,89,69,113,47,55,97,3,37,83,103,27,13,33,115,41,79,17,29,119,75,73,1
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1,85,67,49,125,25,109,43,89,69,113,47,55,97,3,37,83,103,27,13,33,115,41,79,17,29,119,75,73,1
05,7,59,65,21,3,113,61,89,45,107,
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1,85,67,49,125,25,109,43,89,69,113,47,55,97,3,37,83,103,27,13,33,115,41,79,17,29,119,75,73,1
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1,85,67,49,125,25,109,43,89,69,113,47,55,97,3,37,83,103,27,13,33,115,41,79,17,29,119,75,73,1
05,7,59,65,21,3,113,61,89,45,107, /*end 7 */

1,255,79,147,141,89,173,43,9,25,115,97,19,97,197,101,255,29,203,65,195,177,105,17,47,169,109
,149,15,213,135,253,191,155,175,63,47,7,23,39,
1,255,79,147,141,89,173,43,9,25,115,97,19,97,197,101,255,29,203,65,195,177,105,17,47,169,109
,149,15,213,135,253,191,155,175,63,47,7,23,39,
1,255,79,147,141,89,173,43,9,25,115,97,19,97,197,101,255,29,203,65,195,177,105,17,47,169,109
,149,15,213,135,253,191,155,175,63,47,7,23,39,
1,255,79,147,141,89,173,43,9,25,115,97,19,97,197,101,255,29,203,65,195,177,105,17,47,169,109
,149,15,213,135,253,191,155,175,63,47,7,23,39,
1,255,79,147,141,89,173,43,9,25,115,97,19,97,197,101,255,29,203,65,195,177,105,17,47,169,109
,149,15,213,135,253,191,155,175,63,47,7,23,39,
1,255,79,147,141,89,173,43,9,25,115,97,19,97,197,101,255,29,203,65,195,177,105,17,47,169,109
,149,15,213,135,253,191,155,175,63,47,7,23,39,
1,255,79,147,141,89,173,43,9,25,115,97,19,97,197,101,255,29,203,65,195,177,105,17,47,169,109
,149,15,213,135,253,191,155,175,63,47,7,23,39,
1,255,79,147,141,89,173,43,9,25,115,97,19,97,197,101,255,29,203,65,195,177,105,17,47,169,109
,149,15,213,135,253,191,155,175,63,47,7,23,39,
1,255,79,147,141,89,173,43,9,25,115,97,19,97,197,101,255,29,203,65,195,177,105,17,47,169,109
,149,15,213,135,253,191,155,175,63,47,7,23,39,   /*end 8 */

1,257,465,439,177,321,181,225,235,103,411,233,59,353,329,463,385,111,475,451,263,19,249,275,
369,393,167,333,473,469,101,21,451,229,247,297,403,497,61,361,
1,257,465,439,177,321,181,225,235,103,411,233,59,353,329,463,385,111,475,451,263,19,249,275,
369,393,167,333,473,469,101,21,451,229,247,297,403,497,61,361,
1,257,465,439,177,321,181,225,235,103,411,233,59,353,329,463,385,111,475,451,263,19,249,275,
369,393,167,333,473,469,101,21,451,229,247,297,403,497,61,361,
1,257,465,439,177,321,181,225,235,103,411,233,59,353,329,463,385,111,475,451,263,19,249,275,
369,393,167,333,473,469,101,21,451,229,247,297,403,497,61,361,
1,257,465,439,177,321,181,225,235,103,411,233,59,353,329,463,385,111,475,451,263,19,249,275,
369,393,167,333,473,469,101,21,451,229,247,297,403,497,61,361,
1,257,465,439,177,321,181,225,235,103,411,233,59,353,329,463,385,111,475,451,263,19,249,275,
369,393,167,333,473,469,101,21,451,229,247,297,403,497,61,361,
1,257,465,439,177,321,181,225,235,103,411,233,59,353,329,463,385,111,475,451,263,19,249,275,
369,393,167,333,473,469,101,21,451,229,247,297,403,497,61,361,
1,257,465,439,177,321,181,225,235,103,411,233,59,353,329,463,385,111,475,451,263,19,249,275,
369,393,167,333,473,469,101,21,451,229,247,297,403,497,61,361,
1,257,465,439,177,321,181,225,235,103,411,233,59,353,329,463,385,111,475,451,263,19,249,275,
369,393,167,333,473,469,101,21,451,229,247,297,403,497,61,361,
/*end 9 */

1,771,721,1013,759,835,949,113,929,615,157,39,761,169,983,657,647,581,505,833,139,147,203,81
,337,829,989,375,365,131,215,733,451,447,177,57,471,979,197,251,
1,771,721,1013,759,835,949,113,929,615,157,39,761,169,983,657,647,581,505,833,139,147,203,81
,337,829,989,375,365,131,215,733,451,447,177,57,471,979,197,251,
1,771,721,1013,759,835,949,113,929,615,157,39,761,169,983,657,647,581,505,833,139,147,203,81
,337,829,989,375,365,131,215,733,451,447,177,57,471,979,197,251,
1,771,721,1013,759,835,949,113,929,615,157,39,761,169,983,657,647,581,505,833,139,147,203,81
,337,829,989,375,365,131,215,733,451,447,177,57,471,979,197,251,
1,771,721,1013,759,835,949,113,929,615,157,39,761,169,983,657,647,581,505,833,139,147,203,81
,337,829,989,375,365,131,215,733,451,447,177,57,471,979,197,251,
1,771,721,1013,759,835,949,113,929,615,157,39,761,169,983,657,647,581,505,833,139,147,203,81
,337,829,989,375,365,131,215,733,451,447,177,57,471,979,197,251,
1,771,721,1013,759,835,949,113,929,615,157,39,761,169,983,657,647,581,505,833,139,147,203,81
,337,829,989,375,365,131,215,733,451,447,177,57,471,979,197,251,
1,771,721,1013,759,835,949,113,929,615,157,39,761,169,983,657,647,581,505,833,139,147,203,81
,337,829,989,375,365,131,215,733,451,447,177,57,471,979,197,251,
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1,771,721,1013,759,835,949,113,929,615,157,39,761,169,983,657,647,581,505,833,139,147,203,81
,337,829,989,375,365,131,215,733,451,447,177,57,471,979,197,251,
/*end 10 */

1,1285,823,727,267,833,471,1601,1341,913,1725,2021,1905,375,893,1599,415,605,819,975,915,171
5,1223,1367,663,629,525,469,981,1667,1587,1251,451,481,721,483,1209,1457,415,1435,
1,1285,823,727,267,833,471,1601,1341,913,1725,2021,1905,375,893,1599,415,605,819,975,915,171
5,1223,1367,663,629,525,469,981,1667,1587,1251,451,481,721,483,1209,1457,415,1435,
1,1285,823,727,267,833,471,1601,1341,913,1725,2021,1905,375,893,1599,415,605,819,975,915,171
5,1223,1367,663,629,525,469,981,1667,1587,1251,451,481,721,483,1209,1457,415,1435,
1,1285,823,727,267,833,471,1601,1341,913,1725,2021,1905,375,893,1599,415,605,819,975,915,171
5,1223,1367,663,629,525,469,981,1667,1587,1251,451,481,721,483,1209,1457,415,1435,
1,1285,823,727,267,833,471,1601,1341,913,1725,2021,1905,375,893,1599,415,605,819,975,915,171
5,1223,1367,663,629,525,469,981,1667,1587,1251,451,481,721,483,1209,1457,415,1435,
1,1285,823,727,267,833,471,1601,1341,913,1725,2021,1905,375,893,1599,415,605,819,975,915,171
5,1223,1367,663,629,525,469,981,1667,1587,1251,451,481,721,483,1209,1457,415,1435,
1,1285,823,727,267,833,471,1601,1341,913,1725,2021,1905,375,893,1599,415,605,819,975,915,171
5,1223,1367,663,629,525,469,981,1667,1587,1251,451,481,721,483,1209,1457,415,1435,
1,1285,823,727,267,833,471,1601,1341,913,1725,2021,1905,375,893,1599,415,605,819,975,915,171
5,1223,1367,663,629,525,469,981,1667,1587,1251,451,481,721,483,1209,1457,415,1435,
1,1285,823,727,267,833,471,1601,1341,913,1725,2021,1905,375,893,1599,415,605,819,975,915,171
5,1223,1367,663,629,525,469,981,1667,1587,1251,451,481,721,483,1209,1457,415,1435,
/*end 11 */

1,3855,4091,987,1839,4033,2515,579,3863,977,3463,2909,3379,1349,3739,347,387,2881,2821,1873,
1959,1929,2389,3251,1149,243,3609,1131,1701,143,1339,3497,2499,1571,983,4021,1625,3217,1163,
2977,
1,3855,4091,987,1839,4033,2515,579,3863,977,3463,2909,3379,1349,3739,347,387,2881,2821,1873,
1959,1929,2389,3251,1149,243,3609,1131,1701,143,1339,3497,2499,1571,983,4021,1625,3217,1163,
2977,
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1959,1929,2389,3251,1149,243,3609,1131,1701,143,1339,3497,2499,1571,983,4021,1625,3217,1163,
2977,
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2977,
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2977,
1,3855,4091,987,1839,4033,2515,579,3863,977,3463,2909,3379,1349,3739,347,387,2881,2821,1873,
1959,1929,2389,3251,1149,243,3609,1131,1701,143,1339,3497,2499,1571,983,4021,1625,3217,1163,
2977             /*end 12 */
  };
  if (n < 0) {
    for(j=12;j<=MAXBIT;j++) iv[1+j*MAXDIM]=1;  /* Initialize all direction
                                 numbers for the first
                                 coordinate to 1  */
    for (j=1,k=0;j<=MAXBIT;j++,k+=MAXDIM) iu[j] = &iv[k];
    for (k=1;k<=MAXDIM;k++) {
      for (j=1;j<=mdeg[k];j++) iu[j][k] <<= (MAXBIT-j);
      for (j=mdeg[k]+1;j<=MAXBIT;j++) {
  ipp=ip[k];
  i=iu[j-mdeg[k]][k];
  i ^= (i >> mdeg[k]);
  for (l=mdeg[k]-1;l>=1;l--) {
```

```
    if (ipp & 1) i ^= iu[j-1][k];
    ipp >>= 1;
  }
  iu[j][k]=i;
     }
   }
   fac=1.0/(1L << MAXBIT);
   in=0;
 }
 else
   {
     /* Check if the (n-1)-th number was generated in the previous call
   to  sobol. If not, update in and ix  */

     if(in!=n-1) {
  unsigned long gray;

  /* Set ix to 0 */
  for (k=1;k<=IMIN(d,MAXDIM);k++) ix[k]=0;
  in=n-1;
  gray=in^(in>>1);   /* Find gray code of in */
  for (j=1;j<=MAXBIT;j++) {
    if(gray&1) {          /* Only digits which are 1 are used */
      im=(j-1)*MAXDIM;
      for (k=1;k<=IMIN(d,MAXDIM);k++) ix[k] ^= iv[im+k];
    }
    gray>>=1;
  }
     }
     im=in;                  /* Calculate the next vector in the sequence */
     for (j=1;j<=MAXBIT;j++) {   /* Find the rightmost zero bit */
  if (!(im & 1)) break;
  im >>= 1;
     }
     if (j > MAXBIT) nrerror("MAXBIT too small in sobseq");
     im=(j-1)*MAXDIM;
     for (k=1;k<=IMIN(d,MAXDIM);k++) {
  ix[k] ^= iv[im+k];
  x[k-1]=ix[k]*fac;
     }
     in++;
   }
}
#undef MAXBIT
#undef MAXDIM
```

```
/*********************************************************************

  void halton(int n)

   This is the function  halton  for generating Halton points.
   It returns the n-th d-dimensional Halton point.  The point is implicitly
   returned through the array x.  The last two digits of n-1 in base p[j]
   are kept in q1[j] and q2[j].  When both digits become p[j]-1, the
   radical inverse function is computed again.  That way the accumulation
   of round-off error is avoided.  In practice, there are not any upper bounds
   on the values of d and n.

   *********************************************************************/

extern int d;    /* actual dimension of the points */
extern int *q1,*q2;  /* q1[j] is the last digit of n-1 in base p[j], q2[j] is
              the digit before the last one */
extern double *x;     /* This will contain the Halton point */
extern int *p;   /* the first d prime numbers */
extern int *p_1;   /* first d prime numbers minus 1 */
extern double *incr1,*incr2;   /* incr1[j] is 1/p[j] and 1/(p[j]*p[j]) */

double find_fi(int p, int n);  /* See below */

void halton(int n)
{
  double a;
  int j,nn;
  static int ins_n;  /* The default value of ins_n is 0 */

  /* Check if the (n-1)-th number was generated in the previous call
     to  halton.  If not, update q1, q2, and x */

  if(ins_n!=n-1)
    {
     ins_n=n-1;
     for (j=0; j<d; j++)
   {
    q1[j]=ins_n%p[j];
    q2[j]=(ins_n/p[j])%p[j];
    x[j]=find_fi(p[j],ins_n);
   }
     ins_n++;
    }
  for(j=0;j<d;j++)
    {
     if(q1[j]<p_1[j])
   {
    /* It is easy to update when the last digit is less than p[j]-1 */

    q1[j]++;
    x[j]=x[j]+incr1[j];
   }
     else if(q2[j]<p_1[j])
   {

    /* This is the case when the last digit is p[j]-1 and the digit
       before the last one is less than p[j]-1 */

    q1[j]=0;
    q2[j]++;
    x[j]=x[j]+incr1[j]+incr2[j]-1.0;
   }
     else
```

```
    {
      /* This is the case when the last digit is p[j]-1 and the digit
         before the last one is also p[j]-1 */

      q1[j]=0;
      q2[j]=0;
      nn=n/(p[j]*p[j]);
      a=nn%p[j];
      nn=nn/p[j];
      if(nn) x[j]=(a+find_fi(p[j],nn))*incr2[j]*incr1[j];
      else x[j]=a*incr2[j]*incr1[j];
    }
   }
}
/*************************************************************************
  find_fi(int p, int n)

  This returns the radical inverse function fi(p,n) at n for the prime p.

  ************************************************************************/

double find_fi(int p, int n)
{
  int p2,nn;
  double s,fi,incr;

  incr=1.0/p;
  p2=p*p;
  nn=n/p2;

  fi=0.0;
  s=incr;
  while (nn > 0)
    {
      fi+= (nn%p)*s;
      nn=nn/p;
      s*=incr;
    }

  /* The two largest components of fi are added later to avoid possible
     loss of precision */
  fi=((n/p%p)+ fi)/ p2;
  fi+=(n%p)/ (double) p;
  return fi;
}
```

Claims

1. A computer method in financial securities trading, for producing an approximate value for an estimated value of a financial security, comprising:

providing the computer with financial security data;

producing the approximate value by numerically integrating a multivariate integrand whose multi-dimensional integral over a domain of integration represents the estimated value, wherein numerical integration comprises:

evaluating the integrand at points in the domain of integration corresponding to points of a low-discrepancy deterministic sequence, and

combining the integrand values to produce the approximate value; and

producing the approximate value as an output for inspection or/and further processing.

2. The computer method of claim 1, wherein combining the integrand values comprises averaging.

3. The computer method of claim 1, wherein the number of integrand values is predetermined.

4. The computer method of claim 1, wherein the number of integrand values is determined automatically.

5. The computer method of claim 1, further comprising allocating integrand evaluations among a plurality of processors.

6. The computer method of claim 1, further comprising application of an error reduction technique.

7. The computer method of claim 6, wherein error r duction comprises a change of variables.

8. The computer method of claim 6, wherein error reduction comprises variation reduction.

9. The computer method of claim 1, wherein the low-discrepancy deterministic sequence comprises Sobol points.

10. The computer method of claim 1, wherein the low-discrepancy deterministic sequence comprises Halton points.

11. The computer method of claim 1, wherein the low-discrepancy deterministic sequence comprises Hammersley points.

12. The computer method of claim 1, wherein the low-discrepancy deterministic sequence comprises hyperbolic-cross points.

13. The computer method of claim 1, wherein financial securities data comprise derivative instrument data.

14. The computer method of claim 1, further comprising using the approximate value in offering the security for sale.

15. The computer method of claim 1, further comprising using the approximate value in deciding whether to buy, sell or hold the security.

16. A computer system for financial securities trading, for producing an approximate value for an estimated value of a financial security, comprising:

means for providing the computer with financial security data;

means for producing the approximate value by numerically integrating a multivariate integrand whose multi-dimensional integral over a domain of integration represents the estimated value, comprising:

means for evaluating the integrand at points in the domain of integration corresponding to points of a low-discrepancy deterministic sequence, and

means for combining the integrand values to produce the approximate value; and

means for producing the approximate value as an output for inspection or/and further processing.

17. The computer system of claim 16, wherein the means for combining the integrand values comprises averaging means.

18. The computer system of claim 16, wherein the number of integrand values is predetermined.

19. The computer system of claim 16, further comprising means for automatically determining the number of integrand values.

20. The computer system of claim 16, further comprising means for allocating integrand evaluations among a plurality of processors.

21. The computer system of claim 16, further comprising means for applying an error reduction technique.

22. The computer system of claim 21, wherein the means for applying an error reduction technique comprises means for a change of variables.

23. Th computer system of claim 21, wherein the means f r applying an rror reducti n technique compris s means for variation reduction.

24. The computer system of claim 16, wherein the means for producing a low-discrepancy deterministic sequence comprises means for producing Sobol points.

25. The computer system of claim 16, wherein the means for producing a low-discrepancy deterministic sequence comprises means for producing Halton points.

26. The computer system of claim 16, wherein the means for producing a low-discrepancy deterministic sequence comprises means for producing Hammersley points.

27. The computer system of claim 16, wherein the means for producing a low-discrepancy deterministic sequence comprises means for producing hyperbolic-cross points.

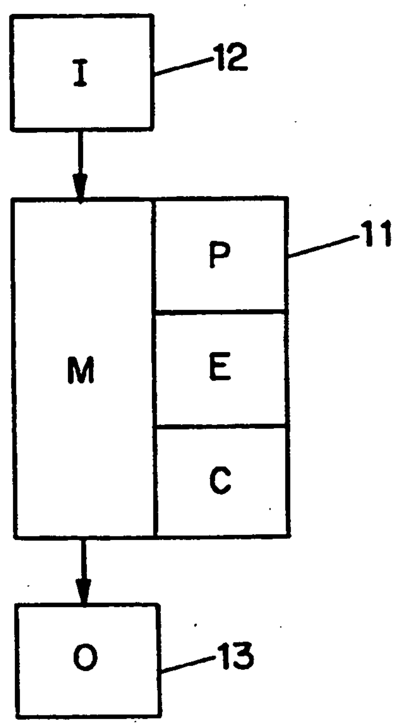
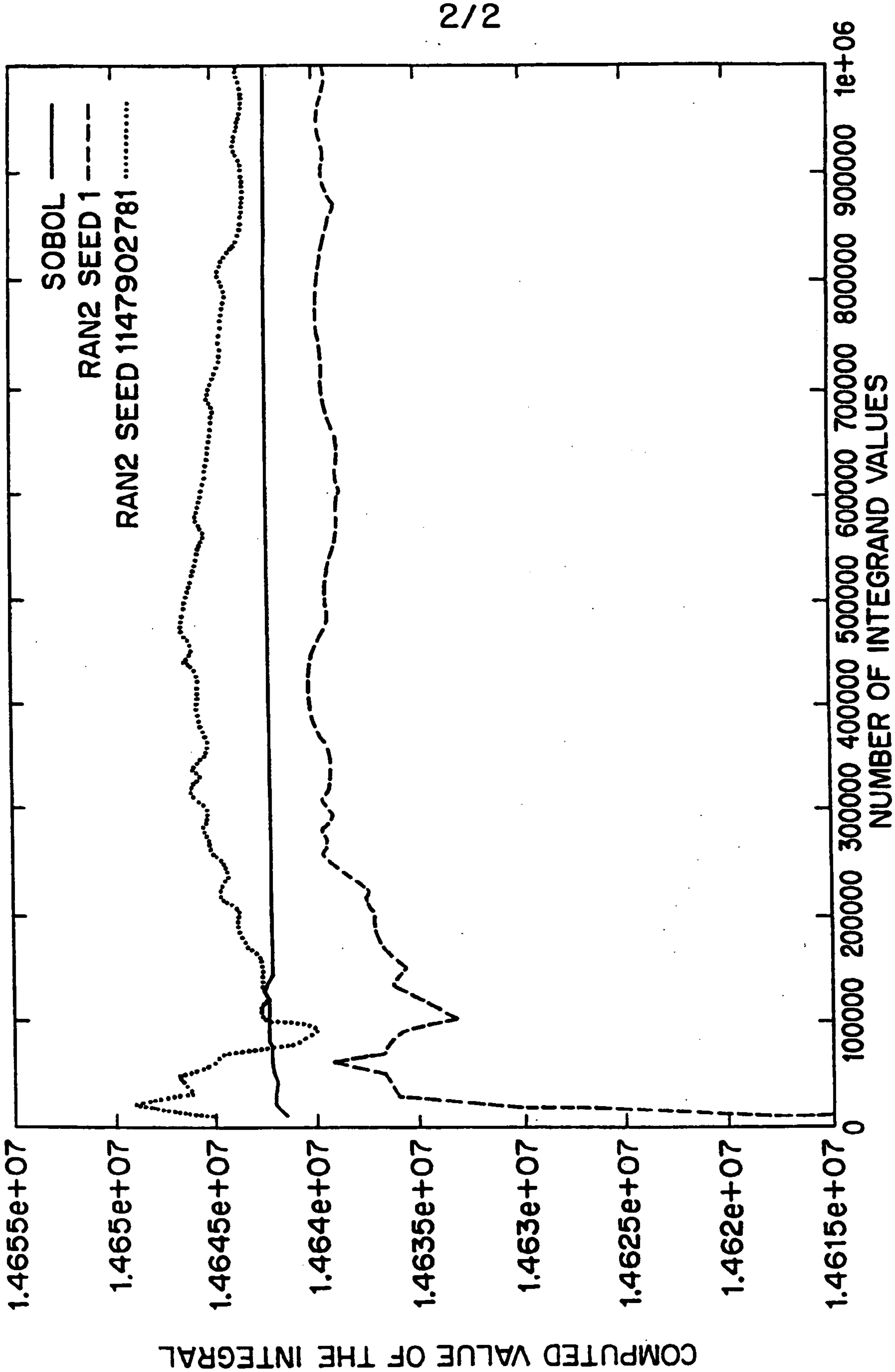


FIG. 1


SOBOL
RAN2 SEED 1
RAN2 SEED 1147902781
COMPUTED VALUE OF THE INTEGRAL
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1.465e+07
1.4645e+07
1.464e+07
1.4635e+07
1.463e+07
1.4625e+07
1.462e+07
1.4615e+07
NUMBER OF INTEGRAND VALUES
0
100000
200000
300000
400000
500000
600000
700000
800000
900000
1e+06

FIG. 2

INTERNATIONAL SEARCH REPORT

International application No.
PCT/US95/10363

A. CLASSIFICATION OF SUBJECT MATTER

IPC(6) :G06F 157:00
US CL :364/408
According to International Patent Classification (IPC) or to both national classification and IPC

B. FIELDS SEARCHED

Minimum documentation searched (classification system followed by classification symbols)

U.S. : 364/408

Documentation searched other than minimum documentation to the extent that such documents are included in the fields searched

Electronic data base consulted during the international search (name of data base and, where practicable, search terms used)

Dialog Database

C. DOCUMENTS CONSIDERED TO BE RELEVANT

Category*	Citation of document, with indication, where appropriate, of the relevant passages	Relevant to claim No.
A	Journal of Complexity, 9, 1993, S. H. Paskov, "Average Case Complexity of Multivariate Integration for Smooth Functions", Pages 291-312	12, 17
Y	Science, V 253, N 5018, 26 July 1991, Barry Cipra, "Mix well, then apply: Math meeting in D.C.", p384(2)	2, 3, 17, 18
A	Monte Carlo Methods, M. Kalos et al, "Monte Carlo Evaluation of Finite-Dimensional Integrals", 1986, pages 89-116	1-27
A	"Random Number Generation and Quasi-Monte Carlo Methods", H. Niederreiter, Society for Applied and Industrial Mathematics, 1992, pages 1-45	1-27

[X] Further documents are listed in the continuation of Box C. [ ] See patent family annex.

* Special categories of cited documents:

"A" document defining the general state of the art which is not considered to be part of particular relevance

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"P" document published prior to the international filing date but later than the priority date claimed

"T" later document published after the international filing date or priority date and not in conflict with the application but cited to understand the principle or theory underlying the invention

"X" document of particular relevance; the claimed invention cannot be considered novel or cannot be considered to involve an inventive step when the document is taken alone

"Y" document of particular relevance; the claimed invention cannot be considered to involve an inventive step when the document is combined with one or more other such documents, such combination being obvious to a person skilled in the art

"&" document member of the same patent family

Date of the actual completion of the international search	Date of mailing of the international search report
13 NOVEMBER 1995	27 DEC 1995

Name and mailing address of the ISA/US	Authorized officer
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INTERNATIONAL SEARCH REPORT

International application No.
PCT/US95/10363

C (Continuation). DOCUMENTS CONSIDERED TO BE RELEVANT

Category*	Citation of document, with indication, where appropriate, of the relevant passages	Relevant to claim No.
Y	Bulletin (New Series) of the American Mathematical Society, Vol. 24, No. 1, January 1991, H. Wozniakowski, "Average Case Complexity of Multivariate Integration", pages 185-194.	1-27
Y	Journal of Complexity 8, 1992, H. Wozniakowski, "Average Case Complexity of Linear Multivariate Problems", pages 373-392	1-27